

A TRIVIAL TAIL HOMOLOGY FOR NON A -ADEQUATE LINKS

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ABSTRACT. We use skein-theoretic techniques to give a simplified proof that the tail of the colored Jones polynomial is trivial for non A -adequate links originally stated in [Lee14]. Translating this proof in homological terms leads to a proof of a conjecture of Rozansky's [Roz14, Conjecture 2.13], which states that the categorification of the colored Jones polynomial of a non A -adequate link has a trivial tail homology.

1. INTRODUCTION

First studied by Lickorish and Thistlethwaite [LT88], *semi-adequate* links form a rich class of links admitting an A -or B -adequate diagram which includes alternating links, see Definition 2.3 and 2.4. Denote the colored Jones polynomial of a link $K \subset S^3$ by $\{J_K(q; n)\}_{n=1}^{\infty}$, where $J_K(q; n) \in \mathbb{Z}[q, q^{-1}]$, and $J_K(q; 1)$ is the unreduced Jones polynomial of K . See Definition 2.2 for the normalization convention we use. Armond [Arm13] and Garoufalidis and Le [GL15] have independently shown the following result. It states a stability property, first conjectured in [DL06, DL07], of the colored Jones polynomial of a semi-adequate link.

Let $d(n)$ be the minimum degree of the n th-colored Jones polynomial $J_K(q; n)$.

Theorem 1.1 ([Arm13, GL15]). *For $i > 1$, let β_i be the coefficient of $q^{d(i)+2(i-2)}$ of $J_K(q; i)$. If K admits an A -adequate diagram, then the coefficient of $q^{d(n)+2(i-2)}$ of $J_K(q; n)$ is equal to β_i for all $n \geq i$.*

For an A -adequate link K one defines a power series

$$(1) \quad T_K(q) = \sum_{i=2}^{\infty} \beta_i q^i,$$

called a *tail* of the colored Jones polynomial of K . For a B -adequate link, we take the mirror image of a B -adequate diagram, and we apply Theorem 1.1 to obtain a *head* of the colored Jones polynomial.

For non semi-adequate links, we have the following theorem, a consequence of which is the generalization of the construction of a tail to all links.

Theorem 1.2 ([Lee14]). *Let $h_n(D)$ be a lower bound of $d(n)$ computed from a link diagram D of K as in (5). Suppose that D is not A -adequate, then*

$$d(n) \geq h_n(D) + 2(n - 1) \text{ for } n > 1.$$

Let $h(n)$ be the maximum of $h_n(D)$ taken over all diagrams D of a link K . This is a link invariant which coincides with $d(n)$ when K is A -adequate. The analogue of Theorem 1.2 for general links is the following corollary.

Corollary 1.3. *For $i > 1$, let β_i be the coefficient of $q^{h(i)+2(i-2)}$ of $J_K(q; i)$. The coefficient of $q^{h(n)+2(i-2)}$ of $J_K(q; n)$ is equal to β_i for all $n \geq i$. When K does not admit an A -adequate link diagram, then $\beta_i = 0$ for all $i > 1$.*

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Let $\widetilde{J}_K(q; n) = q^{-h(n)} J_K(q; n)$ be the n th-shifted colored Jones polynomial. Examples of the stability behavior of Theorem 1.1 and 1.2 are shown in Table 1 and 2.

n	First 6 terms of $\widetilde{J}_K(q, n)$
1	$1 - 2q^2 + 2q^4 - 3q^6 + 3q^8 + 0q^{10} \dots$
2	$\textcircled{1} - 2q^2 - 2q^4 + 7q^6 - 2q^8 - 10q^{10} \dots$
3	$\textcircled{1} \textcircled{-2} q^2 - 2q^4 + 3q^6 + 7q^8 - 2q^{10} \dots$
4	$\textcircled{1} \textcircled{-2} q^2 \textcircled{-2} q^4 + 3q^6 + 3q^8 + 7q^{10} \dots$
5	$\textcircled{1} \textcircled{-2} q^2 \textcircled{-2} q^4 + \textcircled{3} q^6 + 3q^8 + 3q^{10} \dots$

TABLE 1. The stability behavior of an A -adequate link: For the knot 10_{32} , the circled numbers are the identified coefficients as $n \rightarrow \infty$. The coefficients are computed using the KnotTheory package from the Knot Atlas [KAT].

n	First 6 terms of $\widetilde{J}_K(q, n)$
1	$\star + \star q^2 + \star q^4 + \star q^6 + \star q^8 + \star q^{10}$
2	$0 + \star q^2 + \star q^4 + \star q^6 + \star q^8 + \star q^{10}$
3	$0 + 0q^2 + \star q^4 + \star q^6 + \star q^8 + \star q^{10}$
4	$0 + 0q^2 + 0q^4 + \star q^6 + \star q^8 + \star q^{10}$
5	$0 + 0q^2 + 0q^4 + 0q^6 + \star q^8 + \star q^{10}$

TABLE 2. A few first coefficients from $h(n)$ of an arbitrary non A -adequate link. The symbol \star indicates coefficients which are not determined by Theorem 1.2.

We form a tail $J_K(q)$ for any link K by collecting the coefficients β_i in a power series as in (1). Since $d(n) = h_n(D) = h(n)$ when D is an A -adequate diagram of K , the tail $J_K(q)$ coincides with $T_K(q)$ defined in (1) when K is A -adequate.

A consequence of this is that the colored Jones polynomial detects A -adequacy, see [KL14, Lee14, Kal16]. Note that there exist non- A adequate knots, for example the knot $12n706$ [JCC14], for which $d(1) = h(1)$. Indeed, Manchon [Man04] has constructed an infinite family of knots of this type. Thus the Jones polynomial alone does not detect A -adequacy.

Subsequent to [Arm13, GL15], Rozansky [Roz14] has shown that stability behaviors also exist in the categorification of the colored Jones polynomial. More precisely, let $\{H_{i,j}^{Kh}(D; n)\}$ be the set of bigraded chain groups constructed by Rozansky [Roz10] which categorifies the colored Jones polynomial, i.e.,

$$(2) \quad J_K(q; n) = ((-1)^n q^{\frac{n^2+2n}{2}})^{\omega(D)} \sum_{i,j} (-1)^j q^{i+j} \dim H_{i,j}^{Kh}(D; n).$$

See the beginning of Section 2.3 for our grading conventions, and see [FKS06, CK12] for other constructions of the categorification of the polynomial. Let $\widetilde{H}_{*,*}^{Kh}(D; n)$ be the shifted version of the categorifying chain groups whose Euler characteristic is $\widetilde{J}_K(q; n)$. Rozansky shows that there exists a directed system of degree-preserving maps

$$\widetilde{H}^{Kh}(D; n) \xrightarrow{f_n} \widetilde{H}^{Kh}(D; n+1),$$

where f_n are isomorphisms on $\widetilde{H}_{i,*}^{Kh}(D; n)$ for $i \leq n-1$. This implies the existence of a *tail homology* $H^\infty(D)$, which is defined as the direct limit of the directed system. For A -adequate knots, the tail homology categorifies $T_K(q)$.

For non A -adequate knots he makes the following conjecture.

Conjecture 1.4 ([Roz14]). *If a knot is not A -adequate, then $H^\infty(D)$ is trivial.*

Thus Theorem 1.2 gives partial evidence for Conjecture 1.4. The purpose of this paper is to give another proof of Theorem 1.2 and to use it to prove Conjecture 1.4. The main ingredient is a translation of skein-theoretic graphical calculus to the homological calculus of the categorification complex developed by Rozansky. As a result, we obtain a categorification of the tail for all links.

Since analogous statements for non B -adequate links may be obtained by taking the mirror image of a non A -adequate diagram and by considering $J_K(q^{-1}; n)$ instead of $J_K(q; n)$, we will only deal with non A -adequate links for the rest of this paper.

The paper is organized as follows. Section 2 gathers the necessary tools and precise definitions for the main result. We have made an effort for this paper to be self-contained. The definition of the colored Jones polynomial via Jones-Wenzl idempotents is given in Section 2.1, and we review the definition of A -adequate links in Section 2.2. The homological calculus necessary for working with the categorification of the colored Jones polynomial is described in Section 2.3. In Section 3, we give another proof of Theorem 1.2, and Conjecture 1.4 is proven in Section 4 as Theorem 4.1.

2. PRELIMINARIES

2.1. Skein theory and the colored Jones polynomial. We follow the approach of [Lic97]. Let F be an orientable surface which has a finite (possibly empty) collection of points specified on ∂F if $\partial F \neq \emptyset$. A link diagram on F consists of finitely many arcs and closed curves on F such that

- There are finitely many transverse crossings with an over-strand and an under-strand.
- The endpoints of the arcs form a subset of the specified points on ∂F .

Two link diagrams on F are isotopic if they differ by a homeomorphism of F isotopic to the identity. The isotopy is required to fix ∂F .

Definition 2.1. Let q be a fixed complex number. The *linear skein* $\mathcal{S}(F)$ of F is the vector space of formal linear sums over \mathbb{C} of isotopy classes of link diagrams in F quotiented by the relations

$$\begin{aligned} \text{(i)} \quad D \cup \bigcirc &= (-q - q^{-1})D, \\ \text{(ii)} \quad \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} &= q^{1/2} \begin{array}{c} \diagup \\ \diagdown \end{array} \begin{array}{c} \diagdown \\ \diagup \end{array} + q^{-1/2} \begin{array}{c} \diagdown \\ \diagup \end{array} \begin{array}{c} \diagup \\ \diagdown \end{array}. \end{aligned}$$

Note the difference with Rozansky's convention in [Roz14]. His q is q^{-1} here.

We consider the linear skein $\mathcal{S}(D, n)$ of the disc D with $2n$ -points specified on its boundary. For $D_1, D_2 \in \mathcal{S}(D, n)$, there is a natural multiplication operation $D_1 \cdot D_2$ defined by identifying the top boundary of D_1 with the bottom boundary of D_2 . This makes $\mathcal{S}(D, n)$ into an algebra TL_n , called the *Temperley-Lieb algebra*. The algebra TL_n is generated by $1_n, e_n^1, \dots, e_n^{n-1}$, see Figure 1 for an example.

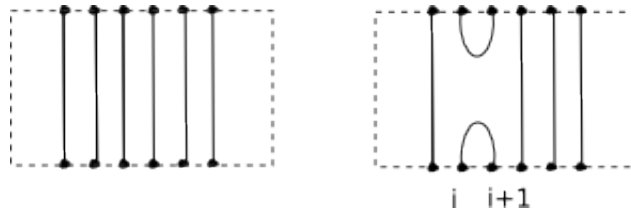


FIGURE 1. An example of the identity element $|_n$ and a generator e_n^i of TL_n for $n = 6$ and $i = 2$.

We will use a shorthand notation which denotes n parallel strands, the identity 1_n , by $|_n$.

Suppose that q^2 is not a k th root of unity for $k \leq n$. There is an element \bigcap_n in TL_n called the *Jones-Wenzl idempotent*, which is uniquely defined by the following properties.

- (i) $\begin{array}{c} \text{---} \\ | \\ \text{---} \end{array}_n \cdot e_n^i = e_n^i \cdot \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array}_n = 0$ for $1 \leq i \leq n-1$.
- (ii) $\begin{array}{c} \text{---} \\ | \\ \text{---} \end{array}_n - 1$ belongs to the algebra generated by $\{e_n^1, e_n^2, \dots, e_n^{n-1}\}$.
- (iii) $\begin{array}{c} \text{---} \\ | \\ \text{---} \end{array}_n \cdot \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array}_n = \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array}_n$,
- (iv) Let $\mathcal{S}(S^1 \times I)$ be the linear skein of the annuli with no points marked on its boundaries. The image of $\begin{array}{c} \text{---} \\ | \\ \text{---} \end{array}_n$ in $\mathcal{S}(S^1 \times I)$ obtained by joining the n boundary points on the top with the those at the bottom is equal to

$$\begin{array}{c} \text{---} \\ | \\ \text{---} \end{array}_n \bigcirc = (-1)^n [n] \cdot \text{the empty diagram on } S^1 \times I,$$

where $[n]$ is the *quantum integer* defined by

$$[n] := \frac{q^{-(n+1)} - q^{n+1}}{q^{-1} - q}.$$

From the defining properties, the Jones-Wenzl idempotent also satisfies a recursion relation and another identity as indicated in the following figures.

$$(3) \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array}_{n+1} = \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array}_n \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array}_1 + \frac{[n-1]}{[n]} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array}_n \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array}_1$$

FIGURE 2. A recursive relation for the Jones-Wenzl projector.

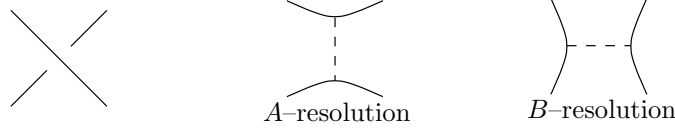
$$(4) \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array}_i \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array}_j = \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array}_{i+j}$$

FIGURE 3. An identity for the Jones-Wenzl projector.

Definition 2.2. Let D be a diagram of a link $K \subset S^3$ with k components. For each component D_i for $i \in \{1, \dots, k\}$ of D take an annuli A_i via the blackboard framing. Let $f : \mathcal{S}(S^1 \times I) \rightarrow \mathcal{S}(\mathbb{R}^2)$ be the map that sends an element of $\mathcal{S}(S^1 \times I)$ to each A_i in the plane. The n th-unreduced colored Jones polynomial $J_K(q; n)$ may be defined as

$$J_K(q; n) := ((-1)^n q^{\frac{n^2+2n}{2}})^{\omega(D)} \left\langle f \left(\begin{array}{c} \text{---} \\ | \\ \text{---} \end{array}_n \right) \right\rangle.$$

The Kauffman bracket here is extended by linearity and gives the polynomial multiplying the empty diagram after reducing the diagram via skein relations. Note that this gives $J_{\bigcirc}(q; 1) = (-1)^n [n]$ as the normalization.

FIGURE 4. A - and B -resolutions of a crossing.

2.2. Semi-adequate links. Let D be a diagram of a link K in S^3 . A *Kauffman state* is a choice of replacing a every crossing of D by the A - or B -resolution as in Figure 4, with the dashed segment recording the location of the crossing before the replacement.

Applying a Kauffman state results in a set of disjoint circles called *state circles*. We form a σ -state graph $s_\sigma(D)$ for each Kauffman state σ by letting the resulting state circles be vertices and the segments be edges. The *all- A state graph* $s_A(D)$ comes from the Kauffman state which chooses the A resolution at every crossing of D .

Definition 2.3. A link diagram D is *A -adequate* if its all- A state graph has no one-edged loops.

A diagram is *B -adequate* if its mirror image is A -adequate.

Definition 2.4. A link K is *semi-adequate* (*A -or B -adequate*) if it admits a diagram that is A -or B -adequate.

We consider the following combinatorial data of an oriented link diagram D and a Kauffman state σ on D .

- $c(D) :=$ The number of crossings in D ,
- $\omega(D) :=$ The writhe of D ,
- $|s_\sigma(D)| :=$ The number of vertices/state circles in the σ -state graph of D .
- $\text{sgn}(\sigma) := \text{sgn}_B(\sigma) - \text{sgn}_A(\sigma)$, where

$\text{sgn}_A(\sigma) := \#$ of crossings where the A -resolution is chosen in σ , and

$\text{sgn}_B(\sigma) := \#$ of crossings where the B -resolution is chosen in σ .

Let

$$(5) \quad h_n(D) = -\frac{n^2}{2}c(D) - n|s_A(D)| + \omega(D)\frac{n^2 + 2n}{2}.$$

We have $d(n) \geq h_n(D)$ and equality is achieved when D is A -adequate [LT88], [Lic97, Lemma 5.4].

2.3. The tail homology of the colored Jones polynomial. The chain complex of the categorification is obtained by composing that of the Jones-Wenzl idempotent, denoted by $\llbracket \bigcap_n \rrbracket$, with the categorification complex $\llbracket D^n \rrbracket$ of the n -blackboard cable of D . Refer to [Roz10] for the details of constructing $\llbracket \bigcap_n \rrbracket$ via the categorification complex of torus braids, and [BN05] for the formal details involved in working with the categorification of tangles. To avoid having to place the double brackets $\llbracket \cdot \rrbracket$ everywhere, a tangle without any bracket denotes its categorification complex. The symbol \sim indicates homotopy equivalence of chain complexes. We are using the same normalization convention as Rozansky [Roz14] except for a substitution of variables—his q is q^{-1} in this paper.

Similar to [Roz14], we have two gradings $i = \deg_h, j = \deg_q$. In this notation, the Khovanov bracket categorifying the Kauffman bracket is given by

- (i) $\llbracket \bigcirc \rrbracket = (\mathbf{q} + \mathbf{q}^{-1})\mathbb{Q}$,
- (ii) $\llbracket \bowtie \rrbracket = \text{Cone} \left(\mathbf{h}^{\frac{1}{2}} \llbracket \bigcap \rrbracket \left(\llbracket \bigcap \rrbracket \xrightarrow{s} \mathbf{h}^{-\frac{1}{2}} \llbracket \bigcup \rrbracket \right) \right),$

with the substitution $\mathbf{h} \rightarrow q$ and $\mathbf{q} \rightarrow -q$, we obtain Khovanov homology. The notation $\text{Cone}(\mathbf{A} \xrightarrow{f} \mathbf{B})$ for two chain complexes $(\mathbf{A}, d_{\mathbf{A}})$ and $(\mathbf{B}, d_{\mathbf{B}})$ indicates the complex of the mapping cone $\mathbf{hA} \oplus \mathbf{B}$ with differential f from $d_{\mathbf{A}}$ and $d_{\mathbf{B}}$. There are two mutually dual categorifications from this approach. For this paper we will use the complex $\llbracket \text{cap}_n \rrbracket$, whose i -grading is bounded from below and therefore has a well-defined minimum degree. This complex has the following universal properties.

Theorem 2.5 ([Roz10, Theorem 2.7]). *Let e_n^i be a generator of TL_n . The complex $\llbracket \text{cap}_n \rrbracket$ has the following properties:*

(i) *The complex of a composition of cap_n with e_n^i for all $i = 1, \dots, n-1$ is contractible, i.e.,*

$$\llbracket e_n^i \circ \text{cap}_n \rrbracket \sim \llbracket \text{cap}_n \circ e_n^i \rrbracket \sim 0.$$

(ii) *The complex $\llbracket \text{cap}_n \rrbracket$ is idempotent with respect to tangle composition, i.e.,*

$$\llbracket \text{cap}_n \circ \text{cap}_n \rrbracket \sim \llbracket \text{cap}_n \rrbracket.$$

We also have the categorified version of (4), which replaces tangles by their categorification complexes and equality by homotopy equivalence [Roz14, Proposition 3.6].

The colored Khovanov bracket categorifying the n th-colored Jones is then obtained by composing the categorification complex of the tangle from the n th-cabled link component, with the complex of the n th-projector. We get a set of bigraded chain groups $\{H_{i,j}^{Kh}(D; n)\}$. The use of the notation \mathbf{h} and \mathbf{q} specifies the changes to the bi-grading completely in what follows.

Note that the colored Khovanov bracket is independent of the framing of the tangle up to a degree shift.

$$(6) \quad \text{cap}_n^a = \mathbf{h}^{-\frac{1}{2}a^2} \mathbf{q}^{-a} \text{cap}_n^a.$$

We define the n th-shifted Khovanov homology.

$$\widetilde{H}^{Kh}(D; n) := \mathbf{h}^{\frac{1}{2}n^2 c(D)} \mathbf{q}^{n|s_A(D)|} H^{Kh}(D; n).$$

Rozansky shows that a tail homology exists for all links.

Theorem 2.6 ([Roz14, Theorem 2.12]). *For any link diagram D , there is a sequence of degree-preserving maps*

$$(7) \quad \widetilde{H}^{Kh}(D; n) \xrightarrow{f_n} \widetilde{H}^{Kh}(D; n+1),$$

which are isomorphisms on $\widetilde{H}^{Kh}_{i,}$ for $i \leq n-1$.*

We give a brief overview of Rozansky's proof for Theorem 2.6 and emphasize key components which we will use in the proof of Conjecture 1.4.

For each n , the degree-preserving map f_n is induced by triples of local skein moves (τ_i, τ_c, τ_f) on link diagrams decorated by the Jones-Wenzl idempotent. The homology groups of the categorification complexes of diagrams (D_i, D_c, D_f) , which differ locally in those skein moves, form a long exact sequence of chain complexes.

$$(8) \quad \mathbf{h}^{-1} H^{Kh}(D'_c) \rightarrow \widetilde{H}^{Kh}(D_f) \xrightarrow{f} \widetilde{H}^{Kh}(D_i) \rightarrow H^{Kh}(D'_c),$$

where $H^{Kh}(D'_c) = \mathbf{h}^{\frac{1}{2}c(D_i)} \mathbf{q}^{|s_A(D_i)|} H^{Kh}(D_c)$.

By (8), we have the desired degree-preserving map $f : \widetilde{H}^{Kh}(D_f, n+1) \rightarrow \widetilde{H}^{Kh}(D_i, n+1)$ if $H_{i,*}^{Kh}(D_c, n+1) = 0$ for suitable i [Roz14, Proposition 3.1]. We list a few triples of local skein

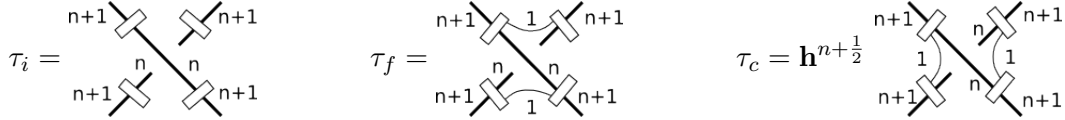


FIGURE 5. Local skein move I. This is a version of the skein relation in Definition 2.1, see also [Yam92].

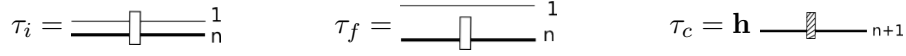


FIGURE 6. Local skein move II. This can be seen from the recursive relation (3) of the Jones-Wenzl idempotent. The complex τ_c comes from the second summand of the recursive relation.

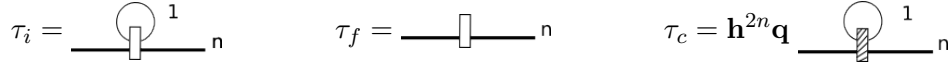


FIGURE 7. Local skein move III. This is similar to local skein move II and the complex τ_c also comes from the second summand of the recursive relation (3).

moves (τ_i, τ_c, τ_f) for which (8) is true, and for which the induced degree-preserving map is an isomorphism for $i \leq n - 1$. Readers interested in the proof may consult [Roz14].

Notice that these sequences of complexes categorify certain moves performed in [Arm13] for A -adequate knots. For Theorem 2.6, Rozansky performs the local skein move I to the link diagram D of K decorated by Jones-Wenzl idempotents with four Jones-Wenzl idempotents framing each crossing, as in the first picture of Figure 5. This results in single-line segments attached at the Jones-Wenzl idempotents in one of the four forms depicted in Figure 8.

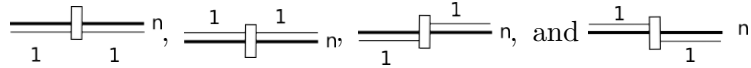


FIGURE 8. Four possibilities after performing the local skein move I at each crossing.

If the two segments on a side of the Jones-Wenzl idempotent are on the same side, then we can pull them off the idempotent via local skein moves II and III. In the end, we have a map from $\widetilde{H^{Kh}}(D; n)$ to $\widetilde{H^{Kh}}(D; n + 1)$ by composing a sequence of degree-preserving maps, which are isomorphisms for $i \leq n - 1$. When the diagram is not A -adequate one can perform a move switching a segment to the opposite side. These moves are illustrated in Figure 9.

For each move switching sides, the original complex is homotopic to the result shifted by $\mathbf{h}^{\pm \frac{n}{2}}$. This cancels out in a pair when we move both ends of a segment to another side. Performing this move so that all the segments are on the same side, we get a curve that can be entirely removed from $\llbracket f \left(\bigcirc_n \right) \rrbracket$.

Example 2.7. We illustrate this on a non A -adequate diagram of two unlinks. See Figure 10, 11.

Definition 2.8. The *tail homology* $H^\infty(D)$ of a link diagram D is the limit of the direct system $\{\widetilde{H^{Kh}}(D; n), f_n\}$ formed by the maps f_n of Theorem 2.5.

$$H^\infty(D) = \lim_{\rightarrow} \widetilde{H^{Kh}}(D; n).$$

By Theorem 2.6, the i -th homology group of $H^\infty(D)$ is isomorphic to $\widetilde{H^{Kh}}_{i-1,*}(D; i + 1)$.

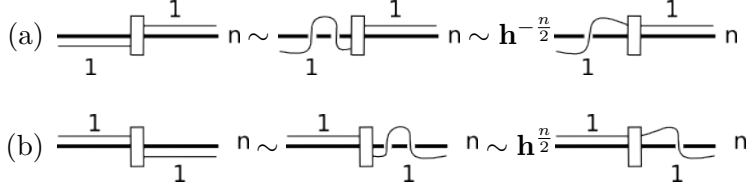
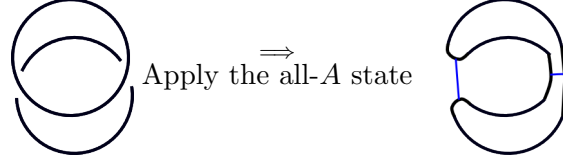
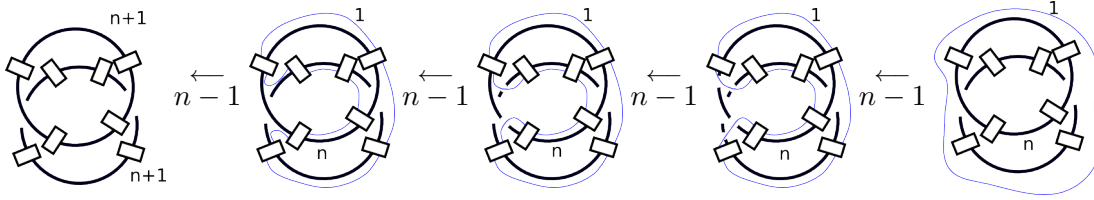


FIGURE 9. One can move a strand over after straightening a curl out.

FIGURE 10. A non A -adequate diagram of two unlinks and the all- A state graph with two one-edged loops.FIGURE 11. The sequence of local moves for which each map $\xleftarrow{n-1}$ is a degree-preserving isomorphism for $i \leq n-1$. At the end, we recover $\widetilde{H^{Kh}}(D; n)$.

3. PROOF OF THEOREM 1.2

We begin by giving another proof of Theorem 1.2. The proof of Conjecture 1.4 will follow from translating the steps in homological terms in Section 4. Throughout this section, we assume that D is not A -adequate, and therefore, it has a one-edged loop in its all A -state graph. Recall that, as in Definition 2.2, we may obtain $J_K(q; n)$ by evaluating the Kauffman bracket on the n -blackboard cable D^n decorated by a Jones-Wenzl idempotent. In our case, we will consider the decoration by four Jones-Wenzl idempotents around a fixed cabled crossing, which corresponds to a loop in the all- A state graph of D , see Figure 12. Without loss of generality, we assume that the loop is attached on the inside of a state circle S of $s_A(D)$. We denote the resulting skein element by D_{\square}^n . The notation for degree, \deg , will always mean the minimum degree of the polynomial or Laurent series in the appropriate variable.

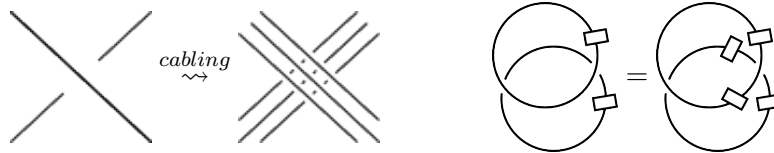


FIGURE 12. On the left: A crossing and its 3-cable. On the right: We put four Jones-Wenzl idempotents around a chosen crossing.

Definition 3.1. Let \mathcal{S} be a skein element with crossings in $\mathcal{S}(\mathbb{R}^2)$ which may or may not be decorated by Jones-Wenzl idempotents. A *generalized Kauffman state* σ is a choice of A - or B -resolution at every crossing of \mathcal{S} . We denote by \mathcal{S}_{σ} the crossingless skein element obtained by

applying σ to the crossings of \mathcal{S} . A σ -state graph $s_\sigma(\mathcal{S})$ is then the set of disjoint circles with segments as before, except for the presence of idempotents.

Let D be a link diagram and consider a skein element \mathcal{S}_σ^n obtained by applying a Kauffman state σ of D^n to D_{\bigoplus}^n . We have the following state sum for the n th-colored Jones polynomial.

$$(9) \quad J_K(q; n) = ((-1)^n q^{\frac{(n^2+2n)}{2}})^{\omega(D)} \sum_{\sigma} q^{\frac{\text{sgn}(\sigma)}{2}} \langle \mathcal{S}_\sigma^n \rangle.$$

In \mathcal{S}_A^n , the circle S cables to n circles S_1, \dots, S_n with at least n loops attached to the inside of the innermost circle S_1 . An example of a generic picture of the all- A state near the cabled crossing is shown in Figure 13.

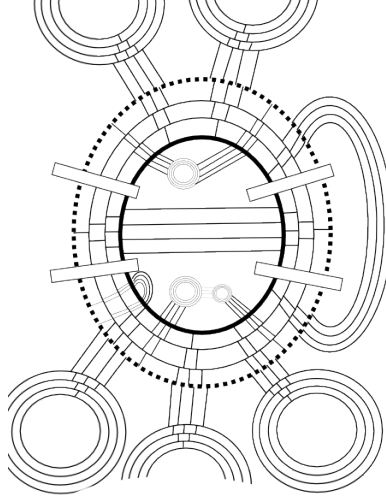


FIGURE 13. A generic local picture for the all- A state graph of a non A -adequate diagram. Here $n = 4$. The thickened inner circle is S_1 and the dashed outer circle is S_n . There may be other one-edged loops connected to S_1 and S_n besides the chosen set, shown as segments framed by four idempotents, and other state circles.

We use the following lemmas from skein theory. These follow from elementary properties (i)-(iv) and (3) of the Jones-Wenzl idempotent.

Lemma 3.2 ([Arm13, Lemma 4]). *Let $\mathcal{S} \in \mathcal{S}(\mathbb{R}^2)$ be a skein element decorated by Jones-Wenzl idempotents \bigoplus_n , and $\overline{\mathcal{S}}$ be the skein element obtained by replacing each Jones-Wenzl idempotent by the identity element $|_n$, then*

$$\deg \langle \mathcal{S} \rangle \geq \deg \langle \overline{\mathcal{S}} \rangle.$$

The next lemma follows immediately from [Lic97, Lemma 5.6] and Lemma 3.2.

Lemma 3.3. *Let $\mathcal{S} \in \mathcal{S}(\mathbb{R}^2)$ be a skein element with crossings decorated by the Jones-Wenzl idempotent \bigoplus_n for some fixed n , and consider the generalized all- A Kauffman state on \mathcal{S} , denoted by \mathcal{S}_A , then*

$$\deg q^{\frac{\text{sgn}(\sigma)}{2}} \langle \overline{\mathcal{S}_\sigma} \rangle \geq \deg q^{\frac{\text{sgn}(A)}{2}} \langle \overline{\mathcal{S}_A} \rangle.$$

Let σ be a generalized Kauffman state for a link diagram D . Lemma 3.3 was implicitly used in [KL14] to show that $\deg \langle D_{\bigoplus}^n \rangle = \deg q^{\frac{\text{sgn}(A)}{2}} \langle \overline{\mathcal{S}_A^n} \rangle$ if and only if D is A -adequate for $n > 1$.

The following lemma is a variant of [Arm13, Lemma 10].

Lemma 3.4. *Let $n - k \geq 1$. We have*

$$\begin{array}{c} \text{Diagram 1} \end{array} = \begin{array}{c} \text{Diagram 2} \end{array} + \frac{[k-1]}{[n-1]} \begin{array}{c} \text{Diagram 3} \end{array}$$

Proof. Similar to the proof of [Arm13, Lemma 10], we apply the recursion relation to the idempotents in the first picture to get

$$(10) \quad \begin{array}{c} \text{Diagram 1} \end{array} = \begin{array}{c} \text{Diagram 2} \end{array} + \frac{[n-2]}{[n-1]} \begin{array}{c} \text{Diagram 3} \end{array}$$

For $n - k > 1$, we apply the recursion formula again to the middle idempotent of the second term of the sum.

$$\begin{array}{c} \text{Diagram 3} \end{array} = \begin{array}{c} \text{Diagram 4} \end{array} + \frac{[n-3]}{[n-2]} \begin{array}{c} \text{Diagram 5} \end{array}$$

It is clear that the first term of the sum above is zero. Sliding the second idempotent to the left in the second term of the sum above by (4), and combining with (10), we see that we get the lemma by repeated expansion via the recursion relation. \square

We make the following definition which generalizes the $\dot{=}_n$ equivalence in [Arm13, Pg. 1].

Definition 3.5. Let s and m be two integers ≥ 0 and $P_1(q)$ and $P_2(q)$ be two Laurent series in q . We write

$$P_1(q) \dot{=}_m^s P_2(q)$$

if and only if the coefficients of $q^s, q^{s+2}, \dots, q^{s+2(m-2)}$ and q 's with power $\leq s$ in $P_1(q)$ agree with those of $P_2(q)$. For example, $2q^1 - q^3 + q^5 \dot{=}_2^1 2q^1 + q^3 + 4q^5$. For two skein elements $\mathcal{S}_1, \mathcal{S}_2$, we write $\mathcal{S}_1 \dot{=}_m^s \mathcal{S}_2$ if $\langle \mathcal{S}_1 \rangle \dot{=}_m^s \langle \mathcal{S}_2 \rangle$.

We consider the following types of skein elements and study their $\dot{=}_n^s$ equivalences.

Definition 3.6. Let D be the diagram of the unknot with one half-twist added and decorated by a Jones-Wenzl idempotent, see Figure 14. Consider the generalized all- A Kauffman state of D_{\square}^n . For $1 \leq j < n$, let $\mathcal{S}^{n,j}$ be the skein element obtained from $s_A(D_{\square}^n)$ by removing all circles and segments outside of \mathcal{S}_j , and replacing each segment by the corresponding crossing before choosing the A -resolution.

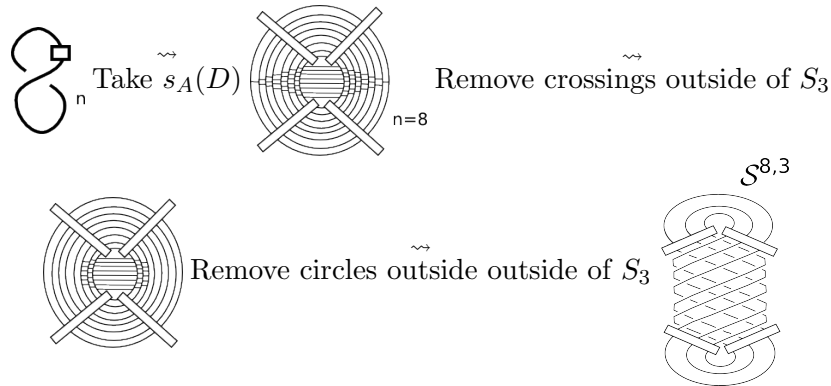


FIGURE 14. An example of $\mathcal{S}^{n,j}$ where $j = 3$ and $n = 8 = 2 \cdot j + 2$.

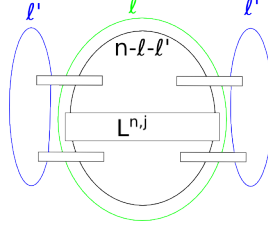


FIGURE 15. The skein element composed of crossings from $\mathcal{S}^{n,j}$ and $\ell + 2\ell'$ circles attached to the idempotents.

Lemma 3.7. Suppose that we have a skein element \mathcal{S} in $\mathcal{S}(\mathbb{R}^2)$ of the form in Figure 15, where $j = n - (\ell + \ell')$, and $L^{n,j}$ is a subset of the crossings of $\mathcal{S}^{n,j}$. Then,

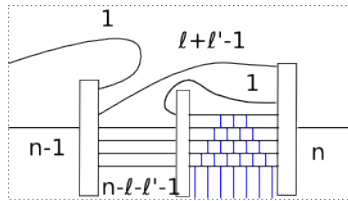
$$(11) \quad \text{Diagram} \stackrel{s}{=}_{n-\ell-\ell'+1} (-q - q^{-1})^{\ell+2\ell'} \text{Diagram},$$

where $s = \deg q^{\frac{\text{sgn}(A)}{2}} \langle \overline{\mathcal{S}_A} \rangle$.

Proof. As in Lemma 3.4, we expand the top idempotent on the right of the first figure in (11).

$$(12) \quad \langle \text{Diagram} \rangle = \underbrace{\langle \text{Diagram} \rangle}_{\mathcal{S}^1} + \frac{[n - \ell - \ell' - 1]}{[n - 1]} \underbrace{\langle \text{Diagram} \rangle}_{\mathcal{S}^2},$$

We write \mathcal{S}^1 for the first term of the sum, and \mathcal{S}^2 for the second. For \mathcal{S}^2 , we have the local picture shown below near the middle idempotent on the right.



In the picture, the vertical/blue segments record the locations of the possible crossings in $L^{n,j}$ prior to applying the all- A Kauffman state. We compare the minimum degree of the Kauffman bracket of \mathcal{S}^1 and \mathcal{S}^2 . Note first that $\deg \langle \mathcal{S}^1 \rangle \geq \deg q^{\frac{\text{sgn}(A)}{2}} \langle \overline{\mathcal{S}_A^1} \rangle = \deg q^{\frac{\text{sgn}(A)}{2}} \langle \overline{\mathcal{S}_A} \rangle = s$. Now

$$\deg \frac{[n - \ell - \ell' - 1]}{[n - 1]} = \ell + \ell'.$$

Let σ be a general Kauffman state on \mathcal{S}^2 , then

$$\langle \mathcal{S}^2 \rangle = \sum_{\sigma} q^{\frac{\text{sgn}(\sigma)}{2}} \langle \mathcal{S}_{\sigma}^2 \rangle.$$

We have the following cases.

- (1) A state σ chooses the A -resolution for all the crossings between S_1 and S_j in the local picture. In this case, we can repeatedly expand the middle idempotent via the recursion relation as in Lemma 3.4 to get that

By an abuse of notation, we will call the skein element with the local picture on the right \mathcal{S}^2 as well. We have $\deg \frac{[0]}{[n-1]} = n-1$, and the number of the circles of \mathcal{S}_{σ}^2 decreases to at least $n-1$ fewer than that of \mathcal{S}_A^1 , from examining how the number of circles in $\overline{\mathcal{S}_{\sigma}^2}$ changes as we apply σ .

Putting these two estimates together gives that $\deg q^{\frac{\text{sgn}(\sigma)}{2}} \langle \mathcal{S}_{\sigma}^2 \rangle \geq \deg q^{\frac{\text{sgn}(A)}{2}} \langle \overline{\mathcal{S}_A^1} \rangle \geq s+2(n-1)$.

- (2) For the other case, we assume that the state σ chooses the B -resolution for some crossing between a pair of state circles S_i and S_{i+1} . Let $k-1$ be the largest such index $i \in \{1, \dots, j-1\}$. Applying Lemma 3.4 repeatedly as before beyond S_k , we have

Now σ has to choose the B -resolution for some crossing between S_i and S_{i+1} for all $i \in \{1, \dots, k-1\}$. Otherwise, there would be a cap or a cup composed with an idempotent on either side that would make $\langle \mathcal{S}_{\sigma}^2 \rangle = 0$. As before, $\deg \frac{[k-1]}{[n-1]} = n-k$, and the condition that σ has to choose the B -resolution for some crossing between S_i and S_{i+1} for all $i \in \{1, \dots, k-1\}$ forces $\deg q^{\frac{\text{sgn}(\sigma)}{2}} \langle \mathcal{S}_{\sigma}^2 \rangle \geq \deg q^{\frac{\text{sgn}(A)}{2}} \langle \overline{\mathcal{S}_A^1} \rangle + 2(n-k) + 2(k-1)$. This is because the number of circles in \mathcal{S}_{σ}^2 decreases by one with each choice of the B -resolution, while $\text{sgn}(\sigma)$ increases by 2. Applying the rest of the Kauffman state σ will not decrease the minimum degree of $\langle \mathcal{S}_{\sigma}^2 \rangle$. This gives that $\mathcal{S} \doteq_n^s \mathcal{S}^1$. Apply this to the rest of the idempotents and keeping track of the change in the number of strands, we get a $\doteq_{n-\ell-\ell'+1}^s$ equivalence of \mathcal{S} with disjoint union of $\ell + 2\ell'$ circles with the component decorated by the idempotents. \square

The key lemma below computes a degree bound on a skein element $\mathcal{S}^{n,j}$ as in Definition 3.6.

Lemma 3.8. *We have for $1 \leq j < n$,*

$$(13) \quad \deg \langle \mathcal{S}^{n,j} \rangle \geq \frac{\text{sgn}(A)}{2} + \deg \langle \overline{\mathcal{S}_A^{n,j}} \rangle + 2j.$$

Proof. Write $n = n'j + r$, so that $r = n \bmod j$. The skein element $\mathcal{S}^{n,j}$ has n' full twists on j strands and a full twist on r strands from examining the braid word and removing crossings with the presence of the projector. Each full twist on j strands will increase the degree by $\frac{j^2}{2} + j$ [Lic97, Lemma 14.1] in addition to removing $n'j^2$ crossings. \square

We now prove

Theorem 3.9. *Suppose that K is not A -adequate, then for $n > 1$,*

$$\deg J_K(q; n) \geq h(n) + 2(n - 1).$$

Let D be a non A -adequate diagram where $h_n(D) = h(n)$. Since $h_n(D) = \deg q^{\frac{\text{sgn}(A)}{2}} \langle \overline{\mathcal{S}}_A^n \rangle$, it suffices to show that

$$\deg \left(\sum_{\sigma} q^{\frac{\text{sgn}(\sigma)}{2}} \langle \mathcal{S}_{\sigma}^n \rangle \right) \geq \deg q^{\frac{\text{sgn}(A)}{2}} \langle \overline{\mathcal{S}}_A^n \rangle + 2(n - 1).$$

Theorem 3.9 follows from the technical lemma below.

Lemma 3.10. *Let D be a non A -adequate diagram with a crossing c corresponding to a loop inside a state circle in its all- A state graph. Let \mathcal{S}_{σ}^n be a skein element obtained by applying a generalized Kauffman state σ to D_{\square}^n , which is decorated with four Jones-Wenzl idempotents at c . Consider σ such that*

$$(14) \quad \deg q^{\frac{\text{sgn}(\sigma)}{2}} \langle \mathcal{S}_{\sigma}^n \rangle < \deg q^{\frac{\text{sgn}(A)}{2}} \langle \overline{\mathcal{S}}_A^n \rangle + 2(n - 1).$$

We have

$$\deg \left(\sum_{\sigma \text{ satisfying (14)}} q^{\frac{\text{sgn}(\sigma)}{2}} \langle \mathcal{S}_{\sigma}^n \rangle \right) \geq \deg q^{\frac{\text{sgn}(A)}{2}} \langle \overline{\mathcal{S}}_A^n \rangle + 2(n - 1).$$

By disregarding these states whose Kauffman brackets have minimum degrees which are too high, we have

$$(15) \quad J_K(q; n) = \sum_{\sigma} q^{\frac{\text{sgn}(\sigma)}{2}} \langle \mathcal{S}_{\sigma}^n \rangle \stackrel{s}{=} \sum_{\sigma \text{ satisfying (14)}} q^{\frac{\text{sgn}(\sigma)}{2}} \langle \mathcal{S}_{\sigma}^n \rangle,$$

with $s = \deg q^{\frac{\text{sgn}(A)}{2}} \langle \overline{\mathcal{S}}_A^n \rangle$. Applying Lemma 3.10 then completes the proof of Theorem 3.9.

Proof. (of Lemma 3.10) We will group the Kauffman states σ satisfying (14) into equivalence classes based on a special property: In applying the state, we cannot merge too many circles from the all- A state, otherwise the lowest degree corresponding to the Kauffman bracket of the state will be too high for it to satisfy (14). This property allows us to compute the degree of the contribution of each equivalence class by comparing it to that from a diagram of the unknot with a half twist added, the skein element $\mathcal{S}^{n,j}$ as in Definition 3.6. Although the sum of the Kauffman brackets of such states will not in general be equal to $\langle \mathcal{S}^{n,j} \rangle$, the first few coefficients relevant to the first $n - 1$ coefficients of $J_K(q; n)$ from s will be, hence they are all 0, giving our statement.

Recall that the state circle S to which the loop corresponding to c is attached in D cables to n circles S_1, \dots, S_n in D^n .

We claim that

- A generalized Kauffman state σ satisfying (14) chooses the A -resolution for crossings corresponding to all the segments between S_i, S_{i+1} in $s_A(D_{\square}^n)$ for some $1 \leq i \leq n - 1$.

To see this, we compare σ to the all- A state. If σ does not choose the A -resolution for crossings corresponding to all the segments between S_i and S_{i+1} for any i , then by Lemma 3.2 and 3.3,

$$\deg q^{\frac{\text{sgn}(\sigma)}{2}} \langle \mathcal{S}_{\sigma}^n \rangle \geq \deg q^{\frac{\text{sgn}(\sigma)}{2}} \langle \overline{\mathcal{S}}_{\sigma}^n \rangle \geq \deg q^{\frac{\text{sgn}(A)}{2}} \langle \overline{\mathcal{S}}_A^n \rangle + 2(n - 1),$$

since $\text{sgn}(\sigma) \geq \text{sgn}(A) + 2(n - 1)$ and choosing the B -resolution for some crossing between each pair of S_i , and S_{i+1} will have merged n circles in $\overline{\mathcal{S}}_{\sigma}^n$ before applying the rest of the state could separate them.

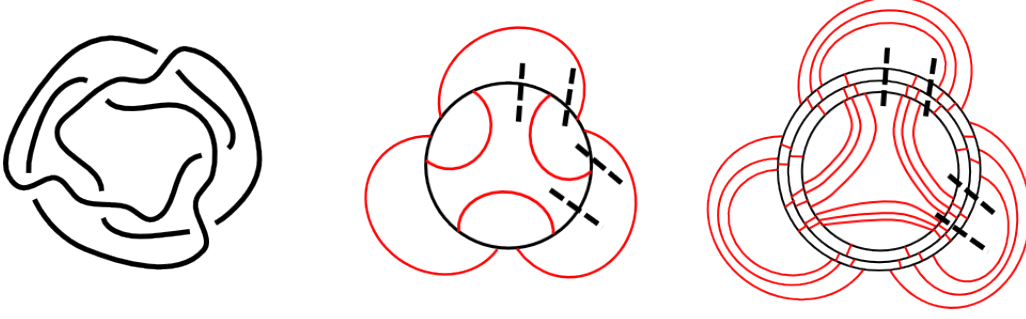


FIGURE 16. An example: The $(3, -3)$ -torus link. The all- A state graph of the diagram shown has 6 loops on a single state circle shown in black. On the very right we have the all- A state graph of the 3-blackboard cable of the diagram. The segments resulting from the Kauffman state is shown in red, and the dash segments are the decorations by four Jones-Wenzl idempotents around a crossing corresponding to a loop.

For σ satisfying (14), let j be the largest i such that σ chooses the A -resolution for all the crossings corresponding to segments between S_i and S_{i+1} for $i \in \{1, \dots, n-1\}$, by the previous claim, we can always find such a j . Let $L^{n,j}$ be the set of crossings corresponding to the cabled loop crossing c inside S_j .

- We define an equivalence relation on the set of generalized Kauffman states satisfying (14).

We say that $\sigma \sim \sigma'$ if and only if

- (a) $j = j'$ for \mathcal{S}_σ^n and $\mathcal{S}_{\sigma'}^n$, respectively.
- (b) σ and σ' are identical outside of $L^{n,j}$.

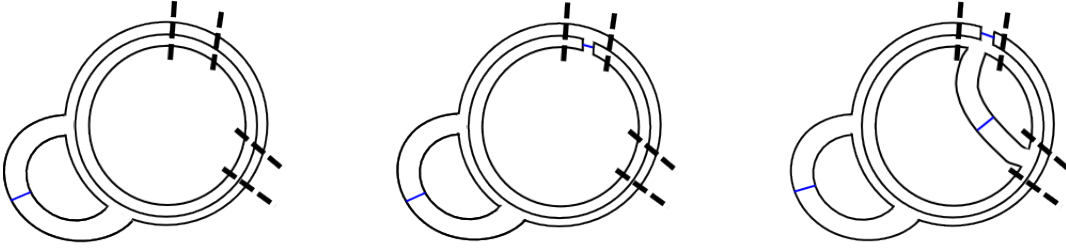


FIGURE 17. Continuing with the example of the $(3,-3)$ -torus link. The first two generalized Kauffman states are in the same equivalence class with $j = 2$, whereas the third one is in a different equivalence class with $j = 1$. The blue segments indicate where the A -resolution has been changed to the B -resolution and the black curves are the new state circles. The red segments where the A -resolution is still chosen have been removed for ease of viewing.

It is clear that \sim is an equivalence relation. In an equivalence class of σ , we may decompose σ as a disjoint union of Kauffman states $\sigma_1 \sqcup \sigma_2$, where σ_2 is restricted to the crossings not in $L^{n,j}$, and σ_1 is restricted to the crossings in $L^{n,j}$. Thus, $\text{sgn}(\sigma) = \text{sgn}(\sigma_1) + \text{sgn}(\sigma_2)$.

We have

$$(16) \quad \sum_{\sigma \text{ satisfying (14)}} q^{\frac{\text{sgn}(\sigma)}{2}} \langle \mathcal{S}_\sigma^n \rangle = \sum_{C \text{ an equivalence class of } \sim} \sum_{\sigma \in C} q^{\frac{\text{sgn}(\sigma)}{2}} \langle \mathcal{S}_\sigma^n \rangle.$$

Since $\text{sgn}(\sigma_2)$ is identical across C , this is equal to

$$(17) \quad = \sum_{C \text{ an equivalence class of } \sim} q^{\frac{\text{sgn}(\sigma_2)}{2}} \sum_{\sigma \in C} q^{\frac{\text{sgn}(\sigma_1)}{2}} \langle \mathcal{S}_\sigma^n \rangle.$$

Let $|\sigma_2|$ be the number of circles disjoint from the component decorated by idempotents after applying σ to crossings not in $L^{n,j}$. We claim the following.

- For an equivalence class C , there exist ℓ, ℓ' such that $j = n - \ell - \ell'$, and

$$(18) \quad \sum_{\sigma \in C} q^{\frac{\text{sgn}(\sigma_1)}{2}} \langle \mathcal{S}_\sigma^n \rangle \stackrel{s_1}{=} (-q - q^{-1})^{|\sigma_2| + \ell + 2\ell'} \langle \mathcal{S}^{n,j} \rangle,$$

where $\mathcal{S}^{n,j}$ is the skein element defined by Definition 3.6, and $s_1 = \frac{\text{sgn}(\sigma_{1A})}{2} + \deg \langle \mathcal{S}_{\sigma_{1A}}^n \rangle$. The skein $\mathcal{S}_{\sigma_{1A}}^n$ is obtained from applying σ_2 in $\sigma = \sigma_1 \sqcup \sigma_2$ and choosing the A -resolution for every crossing in $L^{n,j}$.

For each σ in an equivalence class C , we apply σ_2 , so the crossings of $L^{n,j}$ remain. Each of the original curves S_{j+1}, \dots, S_n in $s_A(D_{\square}^n)$ joins a pair of the four Jones-Wenzl idempotents after applying σ_2 . There are only two possibilities that will not result in a cap or a cup composed with \square_n , which would give 0 for the Kauffman bracket: Either they join the top and bottom two idempotents or the pair on each side. They are indicated by ℓ and ℓ' for the resulting number of closed curves, respectively. Since σ_2 is fixed across C , the numbers ℓ and ℓ' are the same within an equivalence class. We isotope them to the form shown in Figure 15 and apply Lemma 3.7 to remove the circles. This gives the $\stackrel{s_1}{=}$ -equivalence of (18).

Using (16), we have

$$(19) \quad \sum_{\sigma \text{ satisfying (14)}} q^{\frac{\text{sgn}(\sigma)}{2}} \langle \mathcal{S}_\sigma^n \rangle \stackrel{s}{=} \sum_{C \text{ an equivalence class of } \sim} q^{\frac{\text{sgn}(\sigma_2)}{2}} \sum_{\sigma \in C} q^{\frac{\text{sgn}(\sigma_1)}{2}} \langle \mathcal{S}_\sigma^n \rangle,$$

where $s = \frac{\text{sgn}(A)}{2} + \deg \langle \mathcal{S}_A^n \rangle$ as before. Now $\frac{\text{sgn}(\sigma_2)}{2} + s_1 = \deg q^{\frac{\text{sgn}(\sigma_1)}{2}} \langle \mathcal{S}_{\sigma_{1A}}^n \rangle \geq s + 2(n - j - 1)$ for each $\sigma \in C$. This is because at least $\ell + \ell'$ circles would merge through the application of σ_2 where we choose the B -resolution instead of the A -resolution. We have the result after applying Lemma 3.8, where we get that

$$\deg (-q - q^{-1})^{|\sigma_2| + \ell + 2\ell'} \langle \mathcal{S}^{n,j} \rangle \geq s_1 + 2j,$$

so

$$\deg \sum_{\sigma \in C} q^{\frac{\text{sgn}(\sigma_1)}{2}} \langle \mathcal{S}_\sigma^n \rangle \geq s_1 + 2j.$$

□

4. PROOF OF CONJECTURE 1.4

From the definition of the tail homology $H^\infty(D)$, it suffices to show that the homology group $\widetilde{H}^{Kh}_{n-1,*}(D, n+1)$ is trivial. We will show that $\widetilde{H}^{Kh}_{i,*}(D, n+1)$ is trivial for $i \leq n-1$.

Theorem 4.1. *The homology groups $\widetilde{H}^{Kh}_{i,*}(D, n+1)$ are trivial for $i \leq n-1$.*

Proof. We replace the statements in the proof of Theorem 1.2 by their homological versions. The colored Khovanov bracket decomposes into a multi-cone of complexes for each of the generalized Kauffman states obtained by repeatedly applying the colored Khovanov bracket (ii) in Section 2.3.

We see that the chain groups that are relevant for $\widetilde{H}^{Kh}_{i,*}(D, n+1)$ for $i \leq n-1$ is a direct sum of the chain complexes $[\mathcal{S}_\sigma^{n+1}]$ where σ satisfies (14), which may be partitioned into equivalence classes via a decomposition $\sigma = \sigma_1 \sqcup \sigma_2$. Composing with a cap and a cup gives a contractible chain complex by (i) of Theorem 2.5, and so these skeins have the form as described in Figure 15 of Lemma 3.7. For

each equivalence class C with ℓ, ℓ' and $j = n+1-\ell-\ell'$ fixed, we apply local skein moves I-III of Figure 5-7 to obtain a degree-preserving map $f_C : q^{-\ell-2\ell'+|\sigma_2|+\frac{\text{sgn}(\sigma_2)}{2}} \llbracket \widetilde{\mathcal{S}^{n+1,j}} \rrbracket \rightarrow \bigoplus_{\sigma \in C} \mathbf{h}^{\frac{\text{sgn}(\sigma)}{2}} \llbracket \widetilde{\mathcal{S}_\sigma^{n+1}} \rrbracket$ which is an isomorphism for $n-j \leq i \leq n-1$. The \sim indicates complexes that are shifted so that their $\deg_{\mathbf{h}}$ start at the same degree as their image in $\widetilde{H_{i,*}^{Kh}}(D, n+1)$. Note that $\deg_{\mathbf{h}}$ for $\bigoplus_{\sigma \in C} \mathbf{h}^{\frac{\text{sgn}(\sigma)}{2}} \llbracket \widetilde{\mathcal{S}_\sigma^{n+1}} \rrbracket$ is bounded from below by $n-j$.

It suffices to determine the homology groups $\widetilde{H_{i,*}^{Kh}}(\mathcal{S}^{n+1,j})$ for $n-j \leq i \leq n-1$. It was determined in 3.8 that we may remove n' full twists on j strands, which increases $\deg_{\mathbf{h}}$ of the group by at least $n'j^2 \geq j$ by (6). Thus every homology group in $H_{i,*}(\mathcal{S}^{n+1,j})$ for $i \leq n-1$ is trivial. Putting this together with the long exact sequences obtained by applying the colored Khovanov bracket, we see that the homology groups $\widetilde{H_{i,*}^{Kh}}(D, n+1)$ are trivial for $i \leq n-1$. \square

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